

Mermin-Peres quantum magic square and game

Constraint satisfaction problems are abundant in theoretical computer science. Finding solutions or even deciding if they exist can be arduous or even essentially impossible. There are also quantum versions of such problems and in this problem we give an introductory flavor on this topic. We first consider a particular constraint satisfaction problem in the form of a quantum magic square and then introduce a "quantum game".

Classical magic squares are constraint satisfaction problems usually involving constraints satisfied by all the columns and rows in a grid of numbers. For example, suppose we want to construct a 3×3 magic square with numbers in $\{-1, 1\}$ such that their product equals 1 for each row, and equals -1 for each column.

$$M = \left[\begin{array}{c|c|c} a & b & c \\ \hline d & e & f \\ \hline g & h & i \end{array} \right] \quad (1)$$

Question a: *Is there is solution ? Justify.*

A quantum version (analog) of this magic square is a 3×3 grid filled with $n \times n$ hermitian matrices whose eigenvalues are in $\{-1, +1\}$ satisfying the following conditions:

- Given a row, find three hermitian matrices that commute and whose product is the identity matrix.
- Given a column, find three hermitian matrices that commute and whose product is *minus* the identity matrix.

Note that the commutation condition is natural since this way we don't have to worry about the ordering in the products.

Question b: *Check that the following is a solution:*

$$Q = \left[\begin{array}{c|c|c} \sigma_x \otimes \sigma_x & \sigma_x \otimes \mathbf{I} & \mathbf{I} \otimes \sigma_x \\ \hline \sigma_y \otimes \sigma_y & -\sigma_x \otimes \sigma_z & -\sigma_z \otimes \sigma_x \\ \hline \sigma_z \otimes \sigma_z & \mathbf{I} \otimes \sigma_z & \sigma_z \otimes \mathbf{I} \end{array} \right] \quad (2)$$

Remark: Mermin and Peres proved that $n = 4$ is the smallest size of matrices satisfying this quantum magic square.

Now we set up the following *game*. We first go through the *classical* version and after we go through the *quantum* version of the game.

There are two "players" Alice (A) and Bob (B) who can cooperate (communicate) before the game starts to set-up a common strategy. There is a referee also called V for "verifier". Once the game starts A and B cannot communicate. One round of the game is as follows:

- V sends to A the index of a row $i \in \{1, 2, 3\}$ uniformly at random and similarly sends to B the index of a column $j \in \{1, 2, 3\}$ uniformly at random.
- A sends back to V three numbers $a_{i1}, a_{i2}, a_{i3} \in \{-1, +1\}$ such that their product is 1.
- B sends back to V three numbers $b_{1j}, b_{2j}, b_{3j} \in \{-1, +1\}$ such that their product is -1 .
- V checks if $a_{ij} = b_{ij}$ in the intersection of the row i and column j . If this is so A and B win this round of the game. If not, they loose.

The game is played for many rounds and V keeps a record to determine the (empirical) winning probability of A and B.

Question c: *Explain why it is not possible for Alice and Bob to design a classical strategy that always wins this classical game. Design a simple strategy such that Alice and Bob win the game with maximal probability and determine this probability (no formal proof asked).*

Fortunately, Alice and Bob know we don't live in a classical world. Indeed they took quantum science classes in university! Alice and Bob prepare 2 maximally entangled qubits (EPR or Bell pairs) in the state:

$$\begin{aligned} |\psi\rangle_{AB} &= \frac{1}{\sqrt{2}} |B_{00}\rangle_{A_1, B_1} \otimes \frac{1}{\sqrt{2}} |B_{00}\rangle_{A_2, B_2} \\ &= \frac{1}{\sqrt{2}} \left(|00\rangle_{A_1, B_1} + |11\rangle_{A_1, B_1} \right) \otimes \frac{1}{\sqrt{2}} \left(|00\rangle_{A_2, B_2} + |11\rangle_{A_2, B_2} \right) \end{aligned}$$

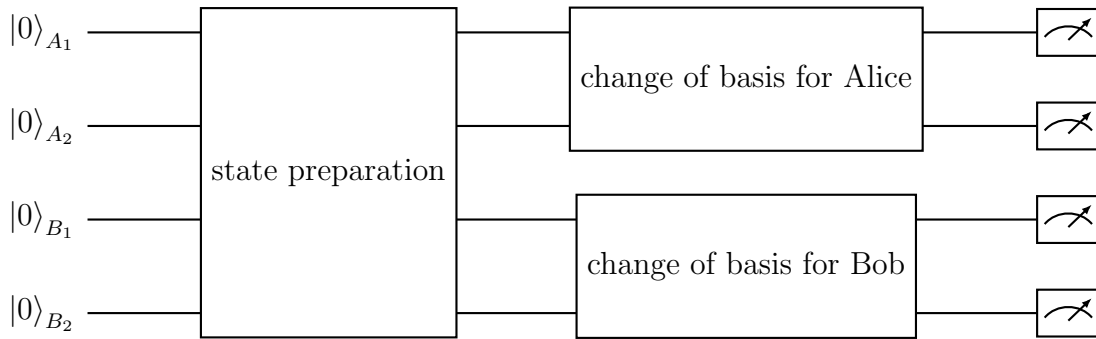
They prepare a big enough reservoir of such states so they can play many rounds of the game (but to fix ideas we think of one round in our subsequent discussion). Their strategy is as follows:

First they prepare and share the state $|\Psi\rangle_{AB}$ before the game starts and they also agree on the quantum magical square Q (equ. (2)). After being given a row i , A makes the measurement described by the observables $Q_{i,1}, Q_{i,2}, Q_{i,3}$, stores the results in a_{i1}, a_{i2}, a_{i3} , and sends these three numbers to V. B proceeds similarly, he stores the results of a simultaneous measurement of the observables $Q_{1,j}, Q_{2,j}, Q_{3,j}$ in b_{1j}, b_{2j}, b_{3j} , and sends the three numbers to V.

Question d: *Explain why A and B win the game for all rounds this time, i.e., with probability equal to 1 (the reminder on the measurement principle at the end might be helpful). Explain this in detail when they receive row $i = 1$ and column $j = 2$ say: find the common measurement basis for the three observables of A, the common measurement basis for the three*

observables of B , check what are the possible outcomes for A and B , and that they must satisfy all conditions for a winning round with probability 1.

Question e: Suppose you want to implement the game on a NISQ device. What are the 9 circuits (of the form below) corresponding to the 9 possible questions of V ?



Question f: for fun for those that know Qiskit. Write a Qiskit code that deals with all possible rows and columns given to Alice and Bob (there are thus 9 possible circuits). The input should be the row i and the column j . The output should be a histogram with the answers of Alice and Bob. Run it on the simulator and then on a real quantum machine.

Reminder of a few facts about the measurement principle:

It is useful to keep in mind the following aspects of the measurement postulate:

1. an observable is a measurable quantity described an hermitian matrix;
2. the measurement apparatus projects the state (or wave function) on one of the eigenbasis vectors $|v\rangle$;
3. the value of the observable is given by the eigenvalue associated with the eigenvector;
4. simultaneous measurements of many observables are only possible for commuting observables since they must have a common eigenbasis.
5. the Born rule states that $\mathcal{P}(|\Psi\rangle \rightarrow |v\rangle) = |\langle v|\Psi\rangle|^2$. If an eigenvalue is degenerate the probability of measuring this eigenvalue is the sum of these probabilities over corresponding eigenvectors.

A remark that you might find useful, is that when eigenvalues are degenerate the corresponding eigenvectors and therefore eigenbasis are not unique.